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Static Cosmological Solutions of the Einstein-Yang-Mills-Higgs Equations

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Abstract

Numerical evidence is presented for the existence of a new family of static, globally regular 'cosmological' solutions of the spherically symmetric Einstein-Yang-Mills-Higgs equations. These solutions are characterized by two natural numbers ($m \geq 1$,$n \geq 0$), the number of nodes of the Yang-Mills and Higgs field respectively. The corresponding spacetimes are static with spatially compact sections with 3-sphere topology.

There has been considerable progress in the study of the spherically symmetric Einstein-Yang-Mills (EYM) and Einstein-Yang-Mills-Higgs (EYMH) equations, stimulated by the discovery of globally regular solutions of the EYM eqs. [1], for a review see e.g. [2]. Up to now most of the attention has been focused on asymptotically flat, smooth particle-like (self-gravitating sphalerons and monopoles) and black hole solutions.

In this paper we present a new 'cosmological' type of globally regular solutions of the EYMH eqs. describing static, spherically symmetric spatially compact space-times. Analogous solutions ('static universes') have already been found in EYM theory [3] *in the presence of a cosmological constant* Λ, (EYMC). More precisely, numerical evidence has been presented in Ref. [3] indicating the existence of a discrete family of solutions indexed by the number m of nodes of the YM field, for a special set of values of the cosmological constant, $\{\Lambda(m),\, m=1,\ldots\}$. The $m=1$ solution is particularly simple as its energy density is constant and it has a simple analytic form [4, 5]. It appears that the presence of the cosmological constant is essential for the very existence of such solutions. Therefore at first sight it might appear surprising that such 'static universe' solutions also exist in an EYMH theory *without a cosmological constant*. In fact this should not come totally unexpected as the self-interaction potential of the scalar field, $V(\Phi)$, can generate the necessary energy density to support a spatially compact space-time. Intuitively $V(\Phi)$ can act as a 'dynamical cosmological constant'. This viewpoint has been particularly stressed by Linde and Vilenkin [6] in their work on 'topological inflation'.

The EYMH theory has three different mass scales, the Planck mass $M_{\rm Pl}=1/\sqrt{G}$ and the masses $M_{\rm W}$ and $M_{\rm H}$ of the YM resp. Higgs field, giving rise to two dimensionless ratios $\alpha=M_{\rm W}/M_{\rm Pl}$ and $\beta=M_{\rm H}/M_{\rm W}$. It turns out to be convenient to introduce an 'effective cosmological constant' $\Lambda=\alpha^4\beta^2/4$. Our $(m=1,n=0,1,2,\ldots)$ EYMH solutions bifurcate with the EYMC solution of [3] for $\alpha=\sqrt{3},\ldots$ at $\Lambda=3/4$ with a vanishing Higgs field. Varying α leads to 1-parameter families $(n=0,1,\ldots)$ with Λ determined as a (possibly multivalued) function of α. Similarly for some discrete values $\alpha_{m,n}$ our solutions bifurcate with the higher nodes cosmological EYMC solutions of Ref. [3], when the Higgs field tends to zero.

Following the notation of [7], we write the spherically symmetric line element as:

$$ds^2=e^{2\nu(R)}dt^2-e^{2\lambda(R)}dR^2-r^2(R)d\Omega^2\,. \tag{1}$$

The 'minimal' spherically symmetric Ansatz for the YM field is

$$W^a_\mu T_a dx^\mu=W(R)(T_1 d\theta+T_2\sin\theta d\varphi)+T_3\cos\theta d\varphi\ , \tag{2}$$

where T_a denote the generators of $SU(2)$ and for the Higgs field

$$\Phi^a=H(R)n^a\ , \tag{3}$$

n^a denoting the unit vector in the radial direction. The reduced EYMH action can be expressed as

$$S=-\int dRe^{(\nu+\lambda)}\Big[\frac{1}{2}\Big(1+e^{-2\lambda}((r')^2+\nu'(r^2)')\Big)-e^{-2\lambda}r^2V_1-V_2\Big]\ , \tag{4}$$

with

$$V_1 = \frac{(W')^2}{r^2} + \frac{1}{2}(H')^2 \,, \tag{5}$$

and

$$V_2 = \frac{(1-W^2)^2}{2r^2} + \frac{\beta^2 r^2}{8}(H^2-\alpha^2)^2 + W^2H^2 \,. \tag{6}$$

Varying the reduced action (4) one obtains the EYMH equations:

$$1 - e^{-2\lambda}\left(r'^2 + \nu'(r^2)'\right) + 2e^{-2\lambda}r^2V_1 - 2V_2 = 0\,, \tag{7a}$$

$$1 + e^{-2\lambda}r'^2 - 2e^{-\lambda}\left(e^{-\lambda}rr'\right)' - 2e^{-2\lambda}r^2V_1 - 2V_2 = 0\,, \tag{7b}$$

$$e^{-\lambda}\left(r'e^{-\lambda}\right)' + e^{-\nu-\lambda}\left(e^{\nu-\lambda}r\nu'\right)' + e^{-2\lambda}\frac{\partial(r^2V_1)}{\partial r} + r\frac{\partial V_2}{\partial r} = 0\,, \tag{7c}$$

$$\left(e^{\nu-\lambda}W'\right)' - e^{\nu+\lambda}W\left(\frac{W^2-1}{r^2} + H^2\right) = 0\,, \tag{7d}$$

$$\left(r^2e^{\nu-\lambda}H'\right)' - e^{\nu+\lambda}H\left(2W^2 + \frac{\beta^2r^2}{2}(H^2-\alpha^2)\right) = 0\,. \tag{7e}$$

Introducing the combinations

$$N \equiv e^{-\lambda}r'\,, \quad \kappa \equiv e^{-\lambda}(r' + r\nu')\,, \tag{8}$$

the field eqs. (7a-e) take the form:

$$2\kappa N = 1 + N^2 + 2\dot{W}^2 + r^2\dot{H}^2 - 2V_2\,, \tag{9a}$$

$$\dot{r} = N\,, \tag{9b}$$

$$\dot{N} = (\kappa - N)\frac{N}{r} - 2\frac{\dot{W}^2}{r} - r\dot{H}^2\,, \tag{9c}$$

$$\dot{\kappa} = [1 - \kappa^2 + 2\frac{\dot{W}^2}{r^2} - \frac{\beta^2r^2}{2}(H^2-\alpha^2)^2 - 2H^2W^2]/r\,, \tag{9d}$$

$$\ddot{W} = W\left(\frac{(W^2-1)}{r^2} + H^2\right) - (\kappa - N)\frac{\dot{W}}{r}\,, \tag{9e}$$

$$\ddot{H} = 2H\left(\frac{W^2}{r^2} + \frac{\beta^2}{4}(H^2-\alpha^2)\right) - (\kappa + N)\frac{\dot{H}}{r}\,. \tag{9f}$$

where $\dot{f} := df/d\sigma = e^{-\lambda}f'$. Note that in Eqs. (9) the remaining gauge freedom (i.e. diffeomorphisms in the variable R) is implicit in the choice of the independent variable σ. A convenient choice (also for the numerical integration) is $e^{\lambda} = 1$.

Next we recall that solutions with a regular origin at $r = 0$ [7] have the expansion

$$W(r) = 1 - br^2 + O(r^4)\,, \tag{10a}$$

$$H(r) = ar + O(r^3)\,, \tag{10b}$$

$$N(r) = 1 - \left(2b^2 + \frac{a^2}{2} + \frac{\alpha^4\beta^2}{24}\right)r^2 + O(r^4)\,, \tag{10c}$$

$$\kappa(r) = 1 + \left(2b^2 - \frac{a^2}{2} - \frac{\alpha^4\beta^2}{8}\right)r^2 + O(r^4)\,, \tag{10d}$$

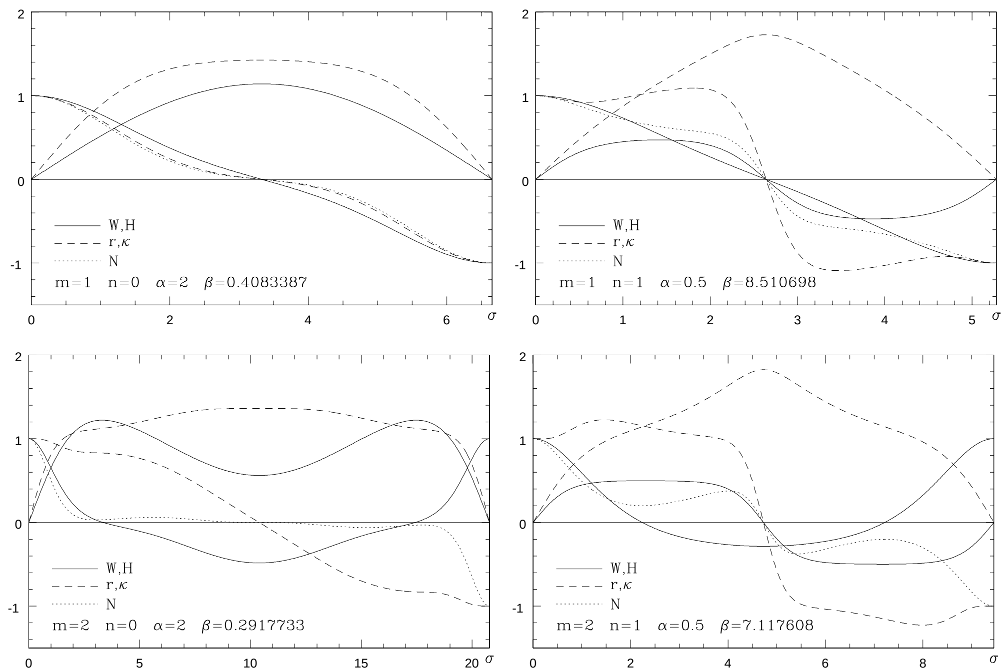

Figure 1: Solutions with $m = 1, 2$ zeros of W and $n = 0, 1$ zeros of H. The figures display W, H (solid), r, κ (dashed), and N (dotted) as functions of σ.

where a, b are free parameters. Solutions satisfying the conditions (10) which also stay globally regular contain (among others) asymptotically flat self-gravitating monopoles. Integrating the field equations (9) one finds that for the generic solution, however, $N(\sigma)$ becomes zero at some finite $\sigma = \sigma_0$ with finite values of all the other dependent variables. As it is immediately seen from Eq. (9b) $N(\sigma_0) = 0$ implies stationarity of r at $r(\sigma_0) = r_0$. Then in general $r(\sigma)$ decreases on some interval for $\sigma > \sigma_0$. We refer to such a point as an 'equator'. In the case of the EYM eqs. it has been proven in Ref. [8] that for all solutions with an equator $r(\sigma)$ decreases from r_0 all the way to $r = 0$ where a curvature singularity develops ('bag of gold'). Adding a cosmological constant radically changes this conclusion and there exists an infinite family of regular solutions with 3-sphere topology [3].

Since for $H(\rho) \equiv 0$ the EYMH system reduces to an EYMC theory where the cosmological constant is given as $\Lambda = \alpha^4\beta^2/4$, it is natural to search for solutions of Eqs. (7) with a nontrivial Higgs field bifurcating with the EYMC ones. To see if such a bifurcating class really exists one should first establish the existence of regular solutions of the linearized Higgs-field equation (7e) in the background of a 3-sphere type solution of the EYMC system since the bifurcation occurs for $H \to 0$. To discuss the linearization of the Higgs-field equation around solutions of the EYMC equations, it is more convenient to choose a different gauge from $e^\lambda = 1$, namely $\nu = \lambda$ and use the variable ρ defined by $dr/d\rho = r' = e^\lambda N$. Then the linearization of the Higgs-field equation (7e), around an EYMC background solution ($\Lambda = \Lambda_b$, $r = r_b(\rho)$, $\nu = \nu_b(\rho)$, $W = W_b(\rho)$, $H = H_b \equiv 0$) with $\Lambda_b = \alpha_b^4\beta_b^2/4$) and $H = h(\rho)$, reads

Table 1: Bifurcation points.

m	Λ_m	$\alpha_{m,0}$	$\alpha_{m,1}$	$\alpha_{m,2}$	$\alpha_{m,3}$	$\alpha_{m,4}$
1	0.75	1.732051	0.707107	0.480384	0.369274	0.301511
2	0.3642442	1.929400	1.080086	0.692265	0.521313	0.420586
3	0.2932176	2.694371	1.331930	0.957678	0.712995	0.569524
4	0.2703275	3.629807	1.686984	1.170945	0.922451	0.739788
5	0.2608951	4.730633	2.061886	1.424943	1.102191	0.912326
10	0.2512791	12.441074	4.080242	2.777478	2.137747	1.745428
20	0.2501165	37.982205	8.256034	5.578608	4.284836	3.494772

as:

$$\left(r_b^2 h'\right)' = 2e^{2\nu_b}\left(W_b^2 - \frac{\Lambda_b}{\alpha^2} r_b^2\right) h\,. \tag{11}$$

Let us consider first the simplest, $m = 1$ EYMC solution [4]:

$$W_b = \cos x\,, \qquad H_b = 0\,, \qquad r_b = \sqrt{2}\sin x\,, \tag{12a}$$

$$N_b = \kappa_b = \cos x\,, \qquad \nu_b = 0\,, \qquad \Lambda_b = 3/4\,, \tag{12b}$$

where $x = \rho/\sqrt{2}$. Then Eq. (11) in the background (12), reduces to the following simple equation:

$$(\sin^2(x) h'(x))' = \left(2\cos^2(x) - \frac{3}{\alpha^2}\sin^2(x)\right) h(x)\,, \tag{13}$$

where now $'$ stands for the derivative with respect to x. We note that Eq. (13) can be transformed to a hypergeometric equation, however, the solutions of interest, i.e. regular on the background geometry and satisfying the condition (10b), can be directly found by the following trigonometric polynomial Ansatz for $h(x)$

$$h_n(x) = \sum_{k=0}^{n} c_k \sin^{2k+1} x\,. \tag{14}$$

One then easily obtains the recursion relation for the coefficients c_k:

$$c_k = \frac{(2k-1)(2k+1) - 2 - 3/\alpha^2}{(2k+2)(2k+1) - 2}\, c_{k-1}\,, \quad k = 1 \ldots n\,, \tag{15}$$

together with the termination condition ($c_{n+1} = 0$)

$$\alpha_{1,n}^2 = \frac{3}{(2n+1)(2n+3) - 2}\,, \quad n = 0, 1, \ldots \tag{16}$$

yielding $\alpha_{1,0}^2 = 3$, $\alpha_{1,1}^2 = 3/13$, $\alpha_{1,2}^2 = 1/11$, ...

That is we have found an infinite family of regular solutions of the linearized Higgs field equation (bounded ‘zero modes’) in the $m = 1$ EYMC background. These ‘zero modes’, indexed by the number of zeros of $h(x)$ ($n = 0, 1, \ldots$), indicate the existence of a family of globally regular solutions of the EYMH

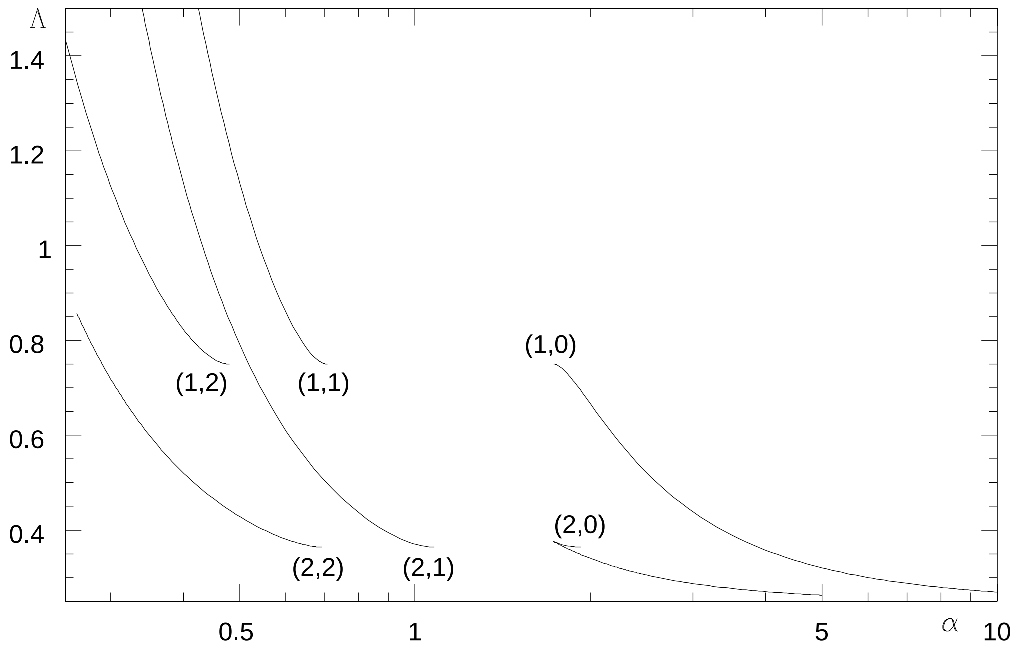

Figure 2: The effective cosmological constant $\Lambda = \alpha^4\beta^2/4$ as a function of α for the solutions with $m, n \leq 2$.

equations (with $H(x)$ having the same number of zeros) bifurcating with the $m = 1$ EYMC solution for $a = 0$ and $\alpha = \alpha_{1,n}$.

From the α-dependence of the r.h.s. of Eq. (13) it follows that the solutions $h(x)$ vanishing at $x = 0$ have $n+1$ zeros for all $\alpha \in [\alpha_{1,n+1}, \alpha_{1,n})$. According to standard theorems on Sturm-Liouville operators this number gives also the number of bound states of the differential operator in Eq. (13). Thus whenever α crosses $\alpha_{1,n}$ from above a new bound state appears, indicated by the existence of a zero mode for $\alpha = \alpha_{1,n}$.

In contrast to the case $m = 1$, the higher node EYMC solutions ($m \geq 2$) are not known analytically. Nevertheless one can still conclude that for *any* of these EYMC backgrounds an infinite family of regular zero modes exists. To see this, it is sufficient to note that for α sufficiently small the r.h.s. of Eq. (11) becomes arbitrarily negative over a finite interval of ρ. This implies that the number of bound states tends to infinity for $\alpha \to 0$. According to the previous argument the same holds for the number of zero modes accumulating at $\alpha = 0$. Numerical values for the first five bifurcation points with $m = 1, \ldots, 5, 10, 20$ are given in Tab. 1.

In order to verify the correctness of the above scenario we have integrated the field eqs. (9) numerically. To simplify the singular boundary value problem, we have looked only for solutions, which are (anti)symmetric about the equator. This means we impose boundary conditions at $r = 0$ and at the equator (a regular point). In addition to the vanishing of κ at the equator, we require the vanishing of the functions W resp. $\dot{W}$ and H resp. $\dot{H}$ depending if m and n are even or odd. Thus 3 functions must have a common zero with N forcing us to tune 3 of the 4 available parameters α, β, a, and b. Fig. 1 shows some solutions

with $m = 1, 2$ zeros of W and $n = 0, 1$ zeros of H.

In Fig. 2 we have plotted the values of the parameters α and Λ for the solutions with $m, n \leq 2$. While α runs to large values with decreasing Λ for the $n = 0$ solutions, i.e. with a nodeless Higgs field, the parameters for the solutions with $n \geq 1$ show the opposite behaviour. This structure remains true for higher values of m, although the change occurs in general for some $n_0 > 1$ increasing with m (e.g. $n_0 = 6$ for $m = 10$). Similar to the case $(2, 0)$ some of the graphs in the α-Λ plane for $m > 2$ show a minimum of α, before they tend to large values of α (e.g. $n = 1, 2, 4, 5$ for $m = 10$).

The families of numerical solutions for which α becomes large seem to have a smooth limit for $\alpha \to \infty$ with $\Lambda \to 1/4$. In this limit, assuming that H stays finite, Eqs. (7) reduce to an EYMH system with a cosmological constant $\Lambda = 1/4$ instead of a Higgs potential ($\beta = 0$). The spatial sections of the corresponding space-times are no longer compact ($\sigma \to \infty$ and $r_0 \to \sqrt{2}$). We have also numerically integrated the limiting field eqs. and our results fully confirm the existence of the limit $\alpha \to \infty$.

It was argued in [11] that the solutions of the EYMC system of [3] are unstable. As usual the criterion for instability is the existence of imaginary modes of the linearized time-dependent field equations. Although we do not quite approve of the methods of [11], we nevertheless believe that their result is correct. For the case $m = 1$ the instability has already been shown in [5]. We expect the instability of the YM system to persist in our case with the Higgs field with the same number of unstable modes at least for small values of a. In view of the discussion of the solutions of Eq. (11) given above there are additional instabilities of the EYMC solutions viewed as solutions of the EYMH theory (with $a = 0$) in the Higgs sector. More precisely, there are $n+1$ unstable modes for $\alpha \in [\alpha_{m,n+1}, \alpha_{m,n})$. Turning to solutions with $a \neq 0$ but small the only questions is, if the number of the unstable Higgs modes of the (m, n) solutions is equal to n or $n + 1$. The answer depends on the behaviour of the zero mode at the bifurcation point $\alpha_{m,n}$ when a deviates from zero. If it turns into a bound state one gets $n + 1$, if it moves into the continuum n unstable modes.

Similar to the globally regular solutions discussed in this paper, there are families of solutions with a horizon branching off from the corresponding solutions of [3] without a Higgs field. In fact there are even more general solutions having two horizons, one of them replacing the regular origin of the forementioned class. We plan to give a detailed account of these solutions in a forthcoming publication.

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